Branched polymers on the Given-Mandelbrot family of fractals

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We study the average number $\overline{A}_n$ per site of the number of different configurations of a branched polymer of n bonds on the Given-Mandelbrot family of fractals using exact real-space renormalization. Different members of the family are characterized by an integer parameter b, $2 \leqslant b \leqslant \infty$. The fractal dimension varies from $\log_2 3$ to 2 as b is varied from 2 to ∞. We find that for all $b \geqslant 3$, $\overline{A}_n$ varies as $\lambda^n \exp(bn^{\psi})$, where λ and b are some constants, and $0 < \psi < 1$. We determine the exponent ψ, and the size exponent ν (average diameter of polymer varies as n^{ν}), exactly for all b, $3 \leqslant b \leqslant \infty$. This generalizes the earlier results of Knezevic and Vannimenus for $b=3$ [Phys. Rev B **35**, 4988 (1987)].

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I. INTRODUCTION

The study of statistical physics models on deterministic fractals has a long history [1–4]. Linear and branched polymers on fractals with a finite ramification number provide very simple and pedagogical examples of renormalization group techniques at work: these system show a nontrivial critical point, and the values of the critical exponents can be determined by linearizing the exact real-space renormalization transformation. The renormalization equations are coupled polynomial recursion equations in a finite number of variables, and are easy to study. By studying different geometrical fractals, one can investigate how the critical exponents change with the geometrical properties of the underlying space.

One particular family of fractals which has been used often for such studies is the Given-Mandelbrot family of fractals [5]. Different members of the family are characterized by an integer b, with $2 \leqslant b \leqslant \infty$. As we increase b from 2 to ∞, the fractal dimension increases from $\log_2 3$ to 2. The critical properties of linear polymers on the $b=2$ fractal were studied in Ref. [2], and these results were extended to $b \leqslant 8$ by Elezovic *et al.* [6] using the exact renormalization equations. Surprisingly, it was found that while the exponent ν_b appeared to converge to the two-dimensional value $3/4$, as b was increased from 2 to 8, the difference in the susceptibility exponent γ_b from the known exact value $43/32$ in two dimensions was found to increase with increasing b. This was explained in Ref. [7], where the asymptotic behavior of critical exponents for large b was determined theoretically using finite-size scaling arguments, and it was shown that γ_b should tend to a different value $133/32$ for large b. Numerical Monte Carlo renormalization group techniques have been used to estimate the critical exponents for significantly larger values of b up to 80 [8,9]. Knezevic and Vannimenus (KV) used the real-space renormalization technique to study the properties of branched polymers on the $b=2$ fractal, and also studied the transition from the extended phase to collapsed phase [10]. This was later extended to other fractals, including the $b=3$ fractal [11]. Dense branched polymers for the $b=2$ fractal have been studied in the context of spanning trees and loop-erased random walks [12], and the Abelian sandpile model [13]. However, a study of branched polymers on fractals for higher b has not been undertaken so far. Nor are the properties of the large-b limit known.

In this paper we study the number of different configurations of an n-bond branched polymer on the Given-Mandelbrot family of fractals using the exact real-space renormalization group techniques. On regular lattices, this number usually varies as $\lambda^n n^{-\theta}$, where λ is some lattice-dependent constant, and θ is a critical exponent. General theoretical arguments that prove the exponential growth would allow stronger correction terms like $\exp(bn^{\psi})$, with $\psi < 1$. Why the first correction term to the exponential growth is a simple power-law term is not fully understood. To see how general is the power-law correction form, one can study this question on different graphs, e.g., fractals. We find the power-law correction also on the $b=2$ fractal. However, this case is exceptional. For all $b \neq 2$, while the number of configurations still increases exponentially with n, the leading correction term to the exponential growth is the stretched-exponential form: this number varies as $\lambda^n e^{bn^{\psi}}$, where λ and b are some constants, and $0 < \psi < 1$. We determine the singularity exponent ψ, and the size exponent ν (average diameter of polymer varies as n^{ν}), exactly for all b, $3 \leqslant b \leqslant \infty$. This generalizes the earlier results of KV for $b=2$ and 3.

This paper is organized as follows: In Sec. II, we start by recapitulating the definition of the Given-Mandelbrot family of fractals, and introduce the generating function for the number of branched polymer configurations with n monomers. Since the fractal does not have translational invariance, we average over different positions of the polymer. The general technique of real-space renormalization applied to these problems is outlined in Sec. III, using the $b=2$ case as an illustrative example. The qualitative behavior of the renormalization equations for $b \geqslant 3$ is discussed in Sec. IV. It turns out that while the equations involve rather complicated high-degree polynomials, the critical exponents ν and ψ do not depend on most of the terms in these polynomials. We can ignore most of these terms, and still determine the *exact* values of these exponents, if we can identify the "dominant terms" in the recursion equations. This is done in Sec. V. Finally, in Sec. VI, using our knowledge of the dominant terms, we determine the exponents ν and ψ for all $b \geqslant 3$,

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